A simple model for aging and the dynamic glass transition: the trap model

We consider a phenomenological model for the dynamics of a spin glass which we will show to reproduce the aging phenomenon typical of these systems. Aging means that the dynamics, or typical response times, get slower the longer the system has been in the glassy phase. Each valley in configuration space (a cluster of low energy configurations that are connected by low energy moves) is represented as a trap, of which there are N in total.

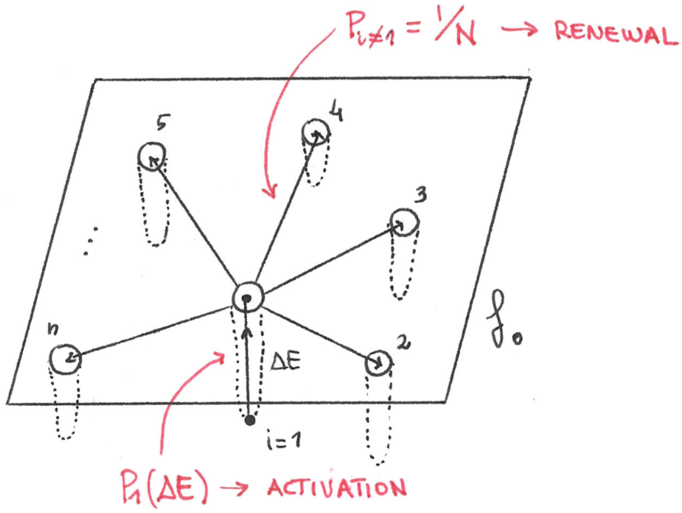

FIG. 1: Schematic representation of the configuration space of the trap model. The threshold free energy f_0 is the *minimal* energy required to go from one metastable state to another one. In general, to reach more distant states, or taking different routes between two states, one may encounter higher threshold values for f. The present model neglects such subtleties. Moreover, there is no geometry or notion of phase space distance between the traps. The traps represent valleys of low energy around a local minimum. These are considered the possible states visited in the course of a stochastic activated evolution in a high-dimensional phase space. The traps are like energy holes, drilled from f_0, their depths representing the free energy difference between the local minimum and the threshold f_0. This model emphasizes the activation/exiting step: the probability to jump to a new trap depends only on the energy of the trap from which the system exits, but not on the energy of the trap into which it falls.

The dynamics is described by jumps from trap to trap. We suppose that once the system reaches a given trap it stays there for a time which is exponentially distributed, with a characteristic time τ_i, that depends on the trap i. This is equivalent to assuming a rate $1/\tau_i$ of exiting the trap i. That is, if the system is still in the trap at time t, the probability of exiting from it between t and $t + dt$ is dt/τ_i. Once the system exits trap i, it jumps towards another trap j, which is chosen uniformly at random among all N traps (for simplicity we assume that one may fall back into the same trap as well). In order to model the heterogeneity of the system and the fact that the dynamics is activated, we suppose that, in order to exit from a trap, an activation energy $E_i > 0$ is necessary to reach a threshold energy where large rearrangements become possible, and that these activation energies are distributed according to

$$P(E) = \frac{1}{E_0} \exp(-E/E_0), \quad E \geq 0. \tag{1}$$

Thus, deeper traps with larger E_i are exponentially less abundant, on the other hand they trap the system for exponentially longer times

$$\tau_i = \tau_0 \exp(E_i/T). \tag{2}$$

Here τ_0 is a microscopic time representing the minimal time to move between very shallow traps. An exponential decrease of the abundance of "metastable states" (configuration valleys) is indeed found in mean field models, especially in structural glasses (while spin glasses have usually a more complex valley structure). It is particularly relevant at temperatures close to the 'dynamical glass temperature' T_d where relatively long-lived metastable configurations (i.e. traps) start appearing. To be more precise, T_d is reached when the resulting dynamics gets slow, which, as we will see, requires that $T \leq E_0 \equiv T_d$. We saw an instance of the exponential dependence of the abundance of states on energy in the REM, where at energies above the ground state there is an exponentially growing number of states. However, we also discussed that dynamics in the REM is trivially super-slow. Here we assume a structure of phase space and a distribution of traps that allows for a more interesting dynamics. In the REM it is just extremely slow whenever one starts at an equilibrium configuration at finite temperature, because the initial state is extensively far in energy from the threshold energy $E = 0$. In the trap model, instead, the initial time to get to the threshold is finite.

1. Derive the distribution of trapping times τ_i, $\rho(\tau)$, from the above distribution of activation energies E_i.
2. After the first escape the system chooses a new trap randomly. What is the expected time it spends in the next trap? What happens for $\mu \equiv T/E_0 \leq 1$, i.e., for $T < E_0$? From your result, argue that the temperature $T_d = E_0$ marks a dynamical glass transition.
3. Write down a differential equation for the time evolution of the probability P_i to find the system in trap i, and find the stationary distribution P_i.
4. Compute the probability that at any given time, when the system has reached a stationary state, the system is found in a trap of escape time τ. What happens in the limit of an infinite number of traps, when $\mu \leq 1$?
5. Consider the dynamical evolution starting from one of the traps chosen uniformly at random at $t = 0$. What is the statistics (mean and variance) of the time elapsed after M jumps,

$$T_M = \sum_{a=1}^{M} \tau_a, \tag{3}$$

in the limit $M \gg 1$? What is the difference between the cases $\mu \leq 1$, $2 \leq \mu > 1$ and $\mu > 2$? Explain why for $\mu \leq 1$ aging is displayed. **Hint:** Determine first how the largest term in the sum scales with M. Then use this as an effective cut-off on $\rho(\tau)$, and estimate mean and variance of the sum for T_M by using the thus truncated $\rho(\tau)$.

For a broader perspective on aging dynamics you may read the review by G. Biroli, https://www.arxiv-vanity.com/papers/cond-mat/0504681/.